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DYNAMICS OF A TWO-LAYER BEAM WITH LAYER SLIP†

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A possible type of structural damping in mechanical systems when there is a distinct dependence of the oscillation decrement on the amplitude is investigated, using the example of the solution of a model problem on the oscillations of a two-layer beam. It is assumed that layer slip only occurs along the beam axis and that the layers move together in the transverse direction. The interaction between the layers is of an elastic-friction form. The equations of motion of the beam in Timoshenko's form are numerically integrated using Godunov's difference scheme. © 2001 Elsevier Science Ltd. All rights reserved.

The oscillations of a two-layer cantilever beam under transverse flexure were investigated in one of the first papers on structural damping [1, 2]. The problem was solved in a quasistatic formulation for the case when a dry friction force, which obeys the Amonton–Coulomb law, acts between the layers.

It is assumed in this paper that the friction forces are associated with a total damping force of the relative slip of the contact surfaces. This approach enables us to describe structural damping when there is a distinct dependence of the oscillation decrement on the amplitude. The free oscillations of a beam, including a beam in which the layers are made of different materials, and the slip of the surfaces at the wave stage of the transients are considered.

1. FORMULATION OF THE PROBLEM

When account is taken of the shear and the rotation inertia of the sections, the motion of the ith layer of the beam is described by the system of equations

$$\begin{aligned}
&\partial N_i / \partial x = \rho_i F_i \partial U_i / \partial t - (-1)^i q, \quad \partial N_i / \partial t = E_i F_i \partial U_i / \partial x \\
&\partial Q_i / \partial x = \rho_i F_i \partial W_i / \partial t, \quad \partial Q_i / \partial t = k_s G_i F_i (\partial W / \partial x - \Phi_i) \\
&\partial M_i / \partial x = \rho_i J_i \partial \Phi_i / \partial t - Q_i + q h_i / 2, \quad \partial M_i / \partial t = E_i J_i \partial \Phi_i / \partial x, \quad i = 1, 2
\end{aligned} \tag{1.1}$$

where N_i, Q_i, M_i are the longitudinal force, the shearing force and the bending moment, U_i, W_i, Φ_i are the longitudinal, transverse and angular velocities of the sections, $E_i G_i$ are the tensile and shear moduli of elasticity, F_i, J_i, h_i, k_s are the area, moment of inertia, height and form factor of the cross-section, ρ_i is the density, q is the interaction force per unit length between the layers, t is the time and x is the axial coordinate.

In Eqs (1.1) we have taken into account the fact that layer slip occurs only along the beam axis and that the layers move together in the transverse direction.

We introduce dimensionless parameters using the relations

$$\begin{aligned}
&N_i = E_0 F_0 N_i^\circ, \quad Q_i = E_0 F_0 Q_i^\circ, \quad M_i = E_0 F_0 A_0 M_i^\circ, \quad U_i = c_0 U_i^\circ \\
&W = c_0 W^\circ, \quad \Phi_i = c_0 \Phi_i^\circ / A_0, \quad E_i = E_0 E_i^\circ, \quad G_i = E_0 G_i^\circ, \quad \rho_i = \rho_0 \rho_i^\circ \\
&F_i = F_0 F_i^\circ, \quad J_i = F_0^2 J_i^\circ, \quad h_i = A_0 h_i^\circ, \quad x = A_0 x^\circ, \quad t = A_0 t^\circ / c_0 \\
&q = E_0 A_0 q^\circ, \quad A_0^2 = F_0, \quad c_0^2 = E_0 / \rho_0
\end{aligned} \tag{1.2}$$

Dimensionless parameters are labelled with the degree superscript and the dimensional quantities used

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in the normalization are labelled with a zero subscript. For brevity, the degree superscript will henceforth be omitted. The equations for the dimensionless parameters are identical to (1.1)

The interaction between the layers has an elastic-friction form

$$q = \psi \operatorname{sign} V + \xi \nu, \quad V \neq 0$$
$$q = q_b + \xi \nu, \quad V = 0 \tag{1.3}$$

where ν and V are the relative displacement of the contacting surfaces and its rate and q_b is the friction force corresponding to complete damping of the relative slip of the surfaces.

We shall assume that the quantity ψ is related to the stress q_b by the power relation

$$\psi = \gamma |q_b|^\alpha \tag{1.4}$$

Relation (1.4) conforms with the power relation between the oscillation decrement and the amplitude, which is characteristic of structural damping and, in the special case when $\alpha = 0$, the Amonton–Coulomb law of dry friction follows from it.

2. SOLUTION

The dynamic processes in the beam were modelled numerically using Godunov's difference scheme [3]. The computational relations for the section of the bar between the cross-sections $x = x_{n-1}$ and $x = x_n$ have the form

$$
\begin{aligned}
&N_i^* = N_{i,*} + E_i F_i (U_{i,n} - U_{i,n-1}) \Delta t / \Delta x \\
&M_i^* = M_{i,*} + E_i J_i (\Phi_{i,n} - \Phi_{i,n-1}) \Delta t / \Delta x \\
&W^* = W_* + (Q_{1,n} + Q_{2,n} - Q_{1,n-1} - Q_{2,n-1}) / (\rho_1 F_1 + \rho_2 F_2) \\
&Q_i^* = (b_1 \rho_i J_i - b_4 k_s G_i F_i \Delta t) / b_3, \quad \Phi_i^* = A_i - B_i q^*, \quad U_i^* = C_i - D_i q^* \\
&b_1 = Q_{i,*} + k_s G_i F_i (W_n - W_{n-1}) \Delta t / \Delta x \\
&b_2 = \rho_i J_i \Phi_{i,*} + (M_{i,n} - M_{i,n-1}) \Delta t / \Delta x, \quad b_3 = \rho_i J_i + k_s G_i F_i \Delta t^2 \\
&b_4 = b_2 - h_i \Delta t q^* / 2, \quad A_i = (b_2 + b_1 \Delta t) / b_3, \quad B_i = h_i \Delta t / (2 b_3) \\
&C_i = U_{i,*} + (N_{i,n} - N_{i,n-1}) \Delta t / (\rho_i F_i \Delta x), \quad D_i = (-1)^{i-1} \Delta t / (\rho_i F_i), \quad i = 1, 2
\end{aligned}
\tag{2.1}
$$

Here i is the number of the layer, the mean integral values of the parameters within the section at t and $t + \Delta t$ are labelled with an asterisk subscript and an asterisk superscript respectively, the subscripts n and $n - 1$ refer to quantities which are mean integral quantities within the limits of Δt for the cross-sections x_n and x_{n-1}, Δt is the time step and Δx is the step along the coordinate.

The total damping force of the relative slip of the layers is given by the formula

$$q_b^* = [C_1 - C_2 + (h_1 A_1 + h_2 A_2)/2] / [D_1 - D_2 + (h_1 B_1 + h_2 B_2)/2] \tag{2.2}$$

To calculate the parameters at the sides x_n and x_{n-1} of a cell of the difference mesh, we use the relations on the characteristics

$$
\begin{aligned}
&N_{i,n} - (j c_{1,i} \rho_i F_i)_{n-j/2} U_{i,n} = (N_i - j c_{1,i} \rho_i F_i U_i)_{n-j/2} \\
&Q_{i,n} - (j c_{2,i} \rho_i F_i)_{n-j/2} W_n = (Q_i - j c_{2,i} \rho_i F_i W)_{n-j/2} \\
&M_{i,n} - (j c_{1,i} \rho_i J_i)_{n-j/2} \Phi_{i,n} = (M_i - j c_{1,i} \rho_i J_i \Phi_i)_{n-j/2} \\
&c_{1,i}^2 = E_i / \rho_i, \quad c_{2,i}^2 = k_s G_i / \rho_i, \quad j = 1, -1
\end{aligned}
\tag{2.3}
$$

where j is the direction cosine of the outer normal to the end of an element of the beam.

At the ends of the beam, one of the quantities occurring in the pairs $(N_{i,n}, U_{i,n})$, $(Q_{i,n}, W_n)$ and $(M_{i,n}, \Phi_{i,n})$ is specified and the other is determined using relations (2.3).

For each time step, a calculation is initially carried out which takes no account of the friction forces, and the quantity sign V, which is used in the subsequent calculation, is determined since the friction forces can only damp the relative motion of the layers but cannot change its direction. In addition, q_b^* is calculated. Equality (2.2) reflects the fact that the momentum of the load q_b^* in a step Δt is sufficient for complete damping of the relative slip of the layers which would be observed at the end of the step if there were no dissipative forces during Δt. The interaction force between the layers and the other parameters of the dynamical process are then determined using (1.3) and (1.4). If it turns out that the rate of relative displacement of the layers nevertheless changes direction compared with the elastic solution, it is assumed that $V = 0$ and the parameters of the process are recalculated for a friction force equal to q_b^*.

The use of simplified relations, which do not contain terms of the order of Δt, for the characteristics and the referral of the non-divergent terms in (2.1) to the moment $t + \Delta t$ increases the margin of stability of the difference scheme. Moreover, a scheme constructed in this manner is only of the first order of approximation. The Runge method [4] was used to increase the order of the approximation to second order. Accordingly, the calculation results were refined using the formula

$$R = 2R(\Delta x) - R(2\Delta x) \tag{2.4}$$

where $R(\Delta x)$, $R(2\Delta x)$ are the values of the parameter R obtained for a single and a double step of the mesh.

The step procedure in [5], which enables one to maintain the Courant numbers equal to unity for all wave systems, was also used to increase the accuracy of the description of the wave phase of the motion and to reduce the mesh viscosity.

This approach and the use of a step size $\Delta x = 0.0025$ enabled us to ensure a fairly low level of mesh viscosity with an acceptable demand on computational resources. Control calculations showed that the decrement, associated with the effects of mesh viscosity, did not exceed 1×10^{-4}.

3. RESULTS OF NUMERICAL CALCULATIONS

The free oscillations of a cantilever beam, damaged in the cross-section $x = 8$, in the case of purely friction interaction between the layers ($\xi = 0$) were initially investigated. The calculations were carried out for a beam consisting of two identical layers $h_i = h$, $E_i = 1$, $\rho_i = 1$. Henceforth, it is assumed everywhere that $k_s = {}^5/_6$, the width of the section $b = 1$ and Poisson's ratio is equal to 0.3. A constant shearing force, which acted during a quarter of a period of the fundamental tone of the transverse oscillations was instantaneously applied to each layer at the end $x = 0$ and then removed. The logarithmic decrement of the oscillations δ was determined using the first ten amplitudes of the oscillations.

The numerical experiment showed that, in the given case, expression (1.4) can be specified in the following way

$$\psi = \beta \, | q_b \Delta x |^{\alpha}, \quad \beta = k_q \frac{|Q_m|^{1-\alpha} \delta}{h} \tag{3.1}$$

Here, Q_m is the actual amplitude of the shearing force in the cross-section of the beam being considered.

It is obvious that $Q_m = k_f k_d$, where k_f is a coefficient which depends on the oscillations mode and k_d is the actual value of the dynamical coefficient. Using the fact that $k_d = |w_m/w_c|$, where w_m is the amplitude of the deflection at the end of the cantilever and w_c is the maximum deflection under static loading, we obtain

$$\beta = k \frac{\delta}{h} \left| \frac{w_m}{w_c} \right|^{1-\alpha}, \quad k = k_q k_f^{1-\alpha} \tag{3.2}$$

It follows from (3.2) that the relations

$$\delta = a k_d^{\alpha-1}, \quad a = \beta h / k = \text{const} \tag{3.3}$$

are satisfied for a fixed value of β in the free oscillation process.

We will now consider the most typical values of α. The case when $\alpha = 0$ corresponds to conventional dry friction, for which the oscillations decrement, as is well known, is inversely proportional to the amplitude. In fact, this dependence also follows from relations (3.3). The change in the deflection w at the end of the beam when $\alpha = 0$ is shown on the left in Fig. 1. The linear attenuation of the amplitude,

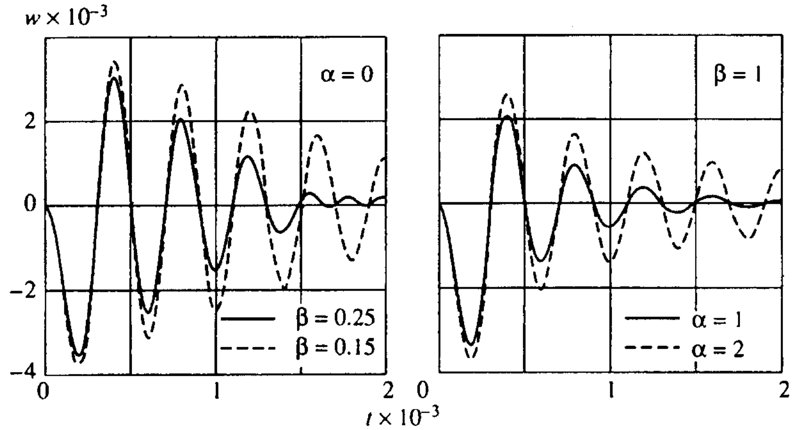

Fig. 1

which is characteristic of dry friction, can be seen. When $\beta = 0.25$ and $t > 1500$, complete cohesion of the contacting surfaces occurs and non-decaying oscillations with an increased frequency are observed.

When $\alpha = 1$, there is damping for which the decrement is independent of the amplitude. The right-hand side of Fig. 1 corresponds to $\beta = 1$. The reduction in the amplitude when $\alpha = 1$ obeys the usual exponential law, and cohesion of the layers is not observed.

Finally, if $\alpha = 2$, it is found that the logarithmic decrement of the oscillations, according to (3.3), is directly proportional to the amplitude. As is shown on the right-hand side of Fig. 1, the decrease in the amplitude at small values of the amplitude is retarded compared with an exponential law, which is indicative of a reduction in the decrement as the amplitude falls.

The change in the parameter a as a function of β for various values of α is shown in Fig. 2. The static deflection at the end of the cantilever when there is no coupling between the layers ($\beta = 0$) was taken as w_c. Over the range under consideration, the magnitude of a depends linearly on β for any α. The dashed line corresponds to the results obtained from a quasistatic solution for a symmetric loading cycle [1, 2] where, naturally, it was assumed that $w_m = w_c$.

An investigation of the free oscillations of a beam, the layers of which differ in their mechanical properties, showed that a reduction in the velocity of sound in one of the layers will lead to a reduction in the oscillations decrement. For example, when $\alpha = 1$ and $\beta = 0.3$, the decrement $\delta = 0.26$ for identical layers. If $c_{1,2} = 0.5c_{1,1}$ then $\delta = 0.20$, and if $c_{1,2} = 0.25c_{1,1}$ then $\delta = 0.12$ for the same values of α and β. It is interesting to note that the result obtained is the same regardless of which characteristic of the material, the modulus of elasticity or the density, changes the velocity of sound.

It has been shown previously in a quasistatic formulation that the efficiency of damping coatings decreases when their modulus of elasticity decreases [6, 7]. The results presented here enable one to ascertain more precisely that it is the velocity of sound in a layer, rather than its modulus of elasticity which is the decisive factor.

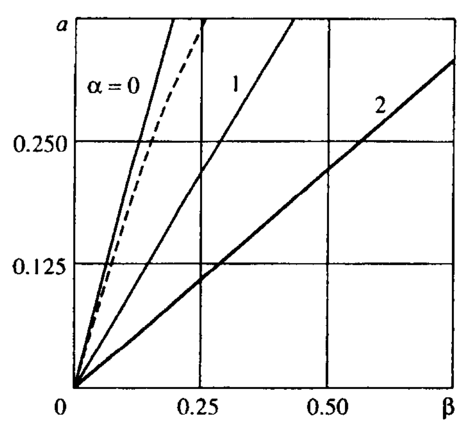

Fig. 2

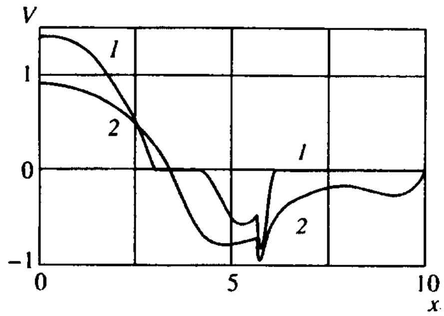

Fig. 3

When $\alpha = 2$, similar changes are observed for the parameter a which, in this case, is equal to δ/k_d. When $\beta = 0.6$ and the layers have identical properties, $a = 0.27$. When $c_{1,2} = 0.5c_{1,1}$, other conditions being equal, $a = 0.16$, and, if $c_{1,2} = 0.25c_{1,1}$, then $a = 0.06$.

When, apart from friction forces, there are also elastic forces of interaction between the layers, this also leads to a reduction in the attenuation of the oscillations. When $\alpha = 1$ and $\beta = 0.3$, then, already when $\xi = 0.01$, the decrement is reduced by a factor of 3.4 and takes the value 0.07. If $\alpha = 2$, a similar reduction is observed in the case of the quantity a which, for the same amplitude, is also equivalent to a decrease in the decrement.

It was assumed in the quasistatic problem that slip begins simultaneously over the whole length of the beam and in a single direction [1, 2]. However, investigation of the transient wave processes shows that slip, at least at the initial wave stage, occurs in a different way. The relative slip rate of the layers of a semi-infinite beam with the same geometrical and mechanical characteristics: $h_i = 1$, $E_i = 1$, $\rho_i = 1$ is shown in Fig. 3 for $t = 10$. When $\alpha = 0$ and $\beta = 0.6$, the travelling domain of cohesion in the middle part of the beam is observable (curve 1). Moreover, there is no slip ahead of the shear wave front. There are no such cohesion zones when $\alpha = 1$ and $\alpha = 2$. Curve 2 corresponds to the slip rate distribution for $\alpha = 2$ and $\beta = 50$. Note that the slip rate has a different direction in different parts of the beam.

The results show that the introduction of a functional and, in particular, a power relation between the friction forces and the overall damping force is an effective method for taking account of the real dependence of structural damping on the amplitude, which is suitable for describing both oscillations and wave processes.

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